

Understanding the U(1) problem through dyon configuration in the Abelian projection

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Abstract

We show that the magnetic monopole promoted to the dyon due to the vacuum angle θ resolves the U(1) problem in the sense that the dyon obtained in this way gives a dominant contribution to the topological susceptibility. For this purpose, we derive an Abelian-projected effective gauge theory written in terms of Abelian degrees of freedom, which is obtained by integrating out all the off-diagonal degrees of freedom involved in the SU(2) Yang-Mills theory with the vacuum angle θ . We evaluate the topological susceptibility by estimating the classical part of the effective dyon action obtained by performing the duality transformation. The obtained result is consistent with the Veneziano–Witten formula.

Key words: U(1) problem, magnetic monopole, dyon, quark confinement,

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1 Introduction

The elementary particles constituting hadrons such as baryons and mesons are called quarks, which are combined by gluons as gauge particles. Nowadays, the very fundamental theory describing quarks and gluons is believed to be quantum chromodynamics (QCD) which is a non-Abelian gauge theory or the Yang-Mills theory with color gauge group $SU(3)$. This is because QCD is the unique theory which well describes the asymptotic behavior of hadrons in high energy region reflecting the asymptotic freedom using the perturbation theory and automatically satisfies a number of conservation laws characteristic in the strong interactions.

On the other hand, the perturbation theory is powerless to study the hadronic phenomena in the low-energy region where the coupling constant becomes large. For instance, it is difficult to calculate the proton mass directly from QCD. Moreover, quarks have never been observed in the isolated form, which is called the quark confinement problem.

There is another problem called the $U(1)$ problem or η meson problem [1]. The η' meson is regarded as a Nambu-Goldstone (NG) boson (pseudo-scalar) associated with the spontaneous breaking of chiral symmetry to the flavor symmetry, $U(N_f)_L \times U(N_f)_R \rightarrow U(N_f)_V$, caused by the flavor-independent quark-antiquark pair condensations $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$. However, η' is very heavy compared with the other eight NG bosons, $\pi^0, \pi^\pm, K^\pm, K^0, \bar{K}^0, \eta$, for $N_f = 3$. In fact, the mass of η' meson is about 958 MeV which is about 10 times larger than the mass 135 MeV of π meson as one of the NG bosons. Moreover, there are inconsistencies between the theoretical prediction and the experimental data for the decay of the η meson, e.g., $\eta \rightarrow \pi^+ \pi^- \pi^0$. These problems have a common origin, i.e., the singularity of the color-flavor singlet axial-vector current (so-called the $U_A(1)$ current).

We know other problems to be resolved, such as strong CP violation and chiral symmetry breaking in the strong interactions. In order to solve these problems, we need to develop the non-perturbative methods without relying on

the perturbation theory. These non-perturbative phenomena are believed to be well understood in the unified way by considering the topologically nontrivial configurations of the gluon field.

The global $U_A(1)$ symmetry is broken at the quantum level, since the $U_A(1)$ current has the triangle anomaly in the quantum theory. In fact, 't Hooft [2] pointed out that topologically nontrivial configurations such as instantons give the nonzero anomaly and suggested that instantons are the relevant topological objects related to the resolution of the $U(1)$ problem [3]. However, it was not clear how to compute the η' mass. Moreover, it was pointed out that the Ward-Takahashi identity for the $U_A(1)$ current with the anomalous term contradicts with the quark-antiquark condensation in the instanton θ vacuum [4].

There is another route initiated by Witten [5] and Veneziano [6] for solving the $U(1)$ problem within the framework of the large N_c (color) expansion. They have derived the relation called the Witten-Veneziano formula which enables us to estimate the η' mass through the topological susceptibility. Along this line, a lot of progress have been made by subsequent works [7]. Nowadays, it is recognized as a solution of the $U(1)$ problem.

In this paper, we argue that the $U(1)$ problem is understood through the dyon configuration. A strategy for solving the $U(1)$ problem along this line has already been discussed by Ezawa and Iwazaki [9] based on the idea of the Abelian projection proposed by 't Hooft [8]. However, they assumed in their analyses the *Abelian dominance* from the beginning and used an Abelian-projected effective theory which is conjectured to be derived from the Yang-Mills theory in the long distance. In contrast, in this paper, *we derive the Abelian-projected effective theory* based on the functional integration of the off-diagonal degrees of freedom from the Yang-Mills theory with the θ angle.

This paper is organized as follows. In section 2, we adopt the BRST formulation to quantize the Yang-Mills theory with the θ angle where we restrict our consideration to the gauge group $SU(2)$ for simplicity. We exploit the Abelian

projection idea [8] and integrate out all the off-diagonal components of gluons. Then we obtain an effective theory written in terms of the diagonal gluons alone, which we call the Abelian Projected Effective Gauge Theory (APEGT) [10] with the θ angle. The Abelian projection here does not mean that the off-diagonal gluons are simply neglected to obtain the APEGT for studying the low-energy physics. In fact, the off-diagonal gluons influence the wavefunction renormalization and the running effective coupling constant in the resulting APEGT to be consistent with the asymptotic freedom in the original Yang-Mills theory [10].

In section 3, we rewrite the APEGT with θ -term into an effective theory written in terms of the dyon degrees of freedom alone. The obtained dyon action has a beautiful form suggesting the existence of the duality in the effective Abelian gauge theory. Here the dyon implies a topological soliton having both electric and magnetic charges where the electric charge of dyon is proportional to the θ angle.

In section 4, we evaluate the topological susceptibility from the effective dyon action. We show that the U(1) problem is solved by using the effective dyon action obtained in this way, if it is combined with the Witten–Veneziano formula.

In the final section, we discuss the relationship between the dyon and the instanton from the viewpoint of understanding the U(1) problem.

2 APEGT with θ term

We extend the method of [10] to the Yang-Mills theory in the presence of the vacuum angle θ .

2.1 Definitions

In this paper, we restrict our consideration to the gauge group $G = SU(2)$. We write the $SU(2)$ gluon field \mathcal{A}_μ as

$$\mathcal{A}_\mu(x) = \sum_{A=1}^3 \mathcal{A}_\mu^A(x) T^A, \quad (1)$$

and the field strength $\mathcal{F}_{\mu\nu}$ as

$$\begin{aligned} \mathcal{F}_{\mu\nu}(x) &= \sum_{A=1}^3 \mathcal{F}_{\mu\nu}^A(x) T^A \\ &= \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) \\ &\quad - ig[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)], \end{aligned} \quad (2)$$

where $T^A (A = 1, 2, 3)$ is the generator of the Lie algebra of the gauge group $SU(2)$. The Hodge dual $\tilde{\mathcal{F}}_{\mu\nu}$ of $\mathcal{F}_{\mu\nu}$ is defined by

$$\tilde{\mathcal{F}}_{\mu\nu}(x) \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathcal{F}_{\alpha\beta}(x). \quad (3)$$

We adopt the Yang-Mills (YM) action $\mathcal{S}_{YM}[\mathcal{A}]$ with the θ term $\mathcal{S}_\theta[\mathcal{A}]$:

$$\mathcal{S}_{YM\theta}[\mathcal{A}] = \mathcal{S}_{YM}[\mathcal{A}] + \mathcal{S}_\theta[\mathcal{A}], \quad (4a)$$

$$\mathcal{S}_{YM}[\mathcal{A}] = -\frac{1}{2g^2} \int_x \text{tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}), \quad (4b)$$

$$\mathcal{S}_\theta[\mathcal{A}] = \frac{\theta}{16\pi^2} \int_x \text{tr}(\mathcal{F}_{\mu\nu} \tilde{\mathcal{F}}^{\mu\nu}), \quad (4c)$$

where we have introduced the notation, $\int_x \equiv \int d^4x$.

The topological term $\mathcal{S}_\theta[\mathcal{A}]$ can be cast into the total derivative and is neglected in the perturbation theory. For the instanton solution with the nontrivial winding number $Q \neq 0$, however, it gives a non-trivial value, $\mathcal{S}_\theta[\mathcal{A}] = \theta Q \neq 0$. Therefore, the topological term is expected to give a non-trivial contribution in the non-perturbative phenomena in which the topological configuration such as instanton plays the important role.

Here, we decompose \mathcal{A}_μ into the diagonal U(1) and the off-diagonal $SU(2)/U(1)$ parts as

$$\begin{aligned} \mathcal{A}_\mu(x) &= a_\mu(x) T^3 + A_\mu(x), \\ A_\mu(x) &:= \sum_{a=1}^2 A_\mu^a(x) T^a, \end{aligned} \quad (5)$$

where the index $a = 1, 2$ denotes the off-diagonal part. $a_\mu(x)$ and $A_\mu^a(x)$ are diagonal, off-diagonal gluon field, respectively. Accordingly, the field

strength $\mathcal{F}_{\mu\nu}$ is decomposed as

$$\mathcal{F}_{\mu\nu} = [f_{\mu\nu}(x) + \mathcal{C}_{\mu\nu}(x)]T^3 + \mathcal{S}_{\mu\nu}^a(x)T^a, \quad (6a)$$

$$f_{\mu\nu}(x) \equiv \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x), \quad (6b)$$

$$\mathcal{S}_{\mu\nu}^a(x) \equiv D_\mu[a]^{ab}A_\nu^b(x) - D_\nu[a]^{ab}A_\mu^b(x), \quad (6c)$$

$$\mathcal{C}_{\mu\nu}(x)T^3 \equiv -i[A_\mu(x), A_\nu(x)], \quad (6d)$$

where the covariant derivative $D_\mu[a]$ is defined by

$$D_\mu[a] = \partial_\mu + i[a_\mu T^3, \cdot], \quad (6e)$$

$$D_\mu[a]^{ab} = \partial_\mu \delta^{ab} - \epsilon^{ab3} a_\mu. \quad (6f)$$

Then the action is decomposed as

$$\mathcal{S}_{YM}[\mathcal{A}] = -\frac{1}{4g^2} \int_x [(f_{\mu\nu} + \mathcal{C}_{\mu\nu})^2 + (\mathcal{S}_{\mu\nu}^a)^2], \quad (7a)$$

$$\mathcal{S}_\theta[\mathcal{A}] = \frac{\theta}{32\pi^2} \int_x [(f_{\mu\nu} + \mathcal{C}_{\mu\nu})(\tilde{f}^{\mu\nu} + \tilde{\mathcal{C}}^{\mu\nu}) + \mathcal{S}_{\mu\nu}^a \tilde{\mathcal{S}}^{\mu\nu a}], \quad (7b)$$

where $(\mathcal{S}_{\mu\nu}^a)^2$ and $\mathcal{S}_{\mu\nu}^a \tilde{\mathcal{S}}^{\mu\nu a}$ are

$$(\mathcal{S}_{\mu\nu}^a)^2 = -2A^{\mu a} W_{\mu\nu}^{ab} A^{\nu b} + 2\partial^\mu (A^{\nu a} \mathcal{S}_{\mu\nu}^a), \quad (7c)$$

$$W_{\mu\nu}^{ab} = (D^\rho[a] D_\rho[a])^{ab} \delta_{\mu\nu} - \epsilon^{ab3} f_{\mu\nu} - D_\mu[a]^{ac} D_\nu[a]^{cb}, \quad (7d)$$

$$\mathcal{S}_{\mu\nu}^a \tilde{\mathcal{S}}^{\mu\nu a} = -2A^{\mu a} \tilde{W}_{\mu\nu}^{ab} A^{\nu b} + 2\partial^\mu (A^{\nu a} \cdot \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{S}^{\sigma\rho}), \quad (7e)$$

$$\tilde{W}_{\mu\nu}^{ab} = -\epsilon_{\mu\nu\alpha\beta} D_\alpha[a]^{ac} D_\beta[a]^{cb} = \epsilon^{ab3} \tilde{f}_{\mu\nu}. \quad (7f)$$

In eq.(7f), we have used

$$[D_\mu[a]^{ac}, D_\nu[a]^{cb}] = -\epsilon^{ab3} f_{\mu\nu} \quad (8)$$

In what follows, the surface terms, i.e., the second terms in (7c) and (7e), are neglected, since it is known that the off-diagonal gluons become massive once the MA gauge fixing is adopted[11].

Thus, the total action $\mathcal{S}_{YM\theta}[\mathcal{A}]$ is decomposed as

$$\begin{aligned} \mathcal{S}_{YM\theta}[\mathcal{A}] = \int_x \left\{ -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \frac{\theta}{32\pi^2} f_{\mu\nu} \tilde{f}^{\mu\nu} \right. \\ \left. + f_{\mu\nu} \left[-\frac{1}{2g^2} \mathcal{C}^{\mu\nu} + \frac{\theta}{16\pi^2} \tilde{\mathcal{C}}^{\mu\nu} \right] \right. \\ \left. - \frac{1}{4g^2} \mathcal{C}_{\mu\nu} \mathcal{C}^{\mu\nu} + \frac{\theta}{32\pi^2} \mathcal{C}_{\mu\nu} \tilde{\mathcal{C}}^{\mu\nu} \right. \\ \left. + \frac{1}{2g^2} A^{\mu a} \left[W_{\mu\nu}^{ab} - \frac{g^2 \theta}{8\pi^2} \epsilon^{ab3} \tilde{f}_{\mu\nu} \right] A^{\nu b} \right\}. \quad (9) \end{aligned}$$

In order to integrate out the off-diagonal gluon field A_μ^a , we replace the terms quartic in A_μ^a in (9),

$$\int_x \left\{ -\frac{1}{4g^2} \mathcal{C}_{\mu\nu} \mathcal{C}^{\mu\nu} + \frac{\theta}{32\pi^2} \mathcal{C}_{\mu\nu} \tilde{\mathcal{C}}^{\mu\nu} \right\}, \quad (10)$$

by the equivalent form quadratic in A_μ^a ,

$$\int_x \left\{ -\frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} B_{\mu\nu} [c_0 \mathcal{C}^{\mu\nu} + c_1 \tilde{\mathcal{C}}^{\mu\nu}] \right\}, \quad (11)$$

with appropriate constants, c_0 and c_1 , to be specified shortly.* In the Minkowski spacetime, by paying attention to the relationship for the double Hodge-dual operations,

$$\tilde{\mathcal{C}}_{\mu\nu} \tilde{\mathcal{C}}^{\mu\nu} = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\alpha\beta} \mathcal{C}^{\rho\sigma} \mathcal{C}_{\alpha\beta} = -\mathcal{C}_{\mu\nu} \mathcal{C}^{\mu\nu}, \quad (13)$$

the Gaussian integration over the $B_{\mu\nu}$ field in (11) is performed to give

$$\begin{aligned} -\frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} B_{\mu\nu} [c_0 \mathcal{C}^{\mu\nu} + c_1 \tilde{\mathcal{C}}^{\mu\nu}] \\ \rightarrow \frac{1}{4g^2} \left\{ (c_0^2 - c_1^2) \mathcal{C}_{\mu\nu} \mathcal{C}^{\mu\nu} + 2c_0 c_1 \mathcal{C}_{\mu\nu} \tilde{\mathcal{C}}^{\mu\nu} \right\}. \quad (14) \end{aligned}$$

Therefore, in order for (10) and (11) to be equivalent, two coefficients, c_0 and c_1 , must satisfy the relationships,[†]

$$c_1^2 - c_0^2 = 1, \quad c_0 c_1 = \frac{g^2 \theta}{16\pi^2}. \quad (15)$$

*This procedure corresponds to introducing the auxiliary (antisymmetric tensor) field $B_{\mu\nu}$ according to

$$B_{\mu\nu} = g^{-2} (c_0 \mathcal{C}_{\mu\nu} + c_1 \tilde{\mathcal{C}}_{\mu\nu}). \quad (12)$$

[†]In the Euclidean space, it should be remarked that the first equation in (15) has the different form, $c_1^2 + c_0^2 = -1$, due to $\tilde{\mathcal{C}}_{\mu\nu} \tilde{\mathcal{C}}^{\mu\nu} = \mathcal{C}_{\mu\nu} \mathcal{C}^{\mu\nu}$.

The solution of (15) is

$$c_0^2 = \frac{1}{2} \left(-1 \pm \frac{g^2}{4\pi} |\tau| \right), \quad c_1^2 = \frac{1}{2} \left(1 \pm \frac{g^2}{4\pi} |\tau| \right), \quad (16)$$

where τ is the complex coupling constant defined by

$$\tau \equiv \frac{\theta}{2\pi} + i \frac{4\pi}{g^2}, \quad |\tau| = \sqrt{\left(\frac{\theta}{2\pi} \right)^2 + \left(\frac{4\pi}{g^2} \right)^2}, \quad (17)$$

which is known to play the very important role especially in the supersymmetric Yang-Mills theory, see e.g., [12].

In what follows, we adopt

$$c_0 = \sqrt{\frac{1}{2} \left(-1 + \sqrt{1 + \left(\frac{g^2 \theta}{8\pi^2} \right)^2} \right)}, \quad c_1 = \sqrt{\frac{1}{2} \left(1 + \sqrt{1 + \left(\frac{g^2 \theta}{8\pi^2} \right)^2} \right)}. \quad (18)$$

For $g^2 \theta \ll 1$, we see the coefficients behave as

$$c_0 \simeq \frac{1}{2} \cdot \frac{g^2 \theta}{8\pi^2}, \quad c_1 \simeq 1 + \frac{1}{8} \left(\frac{g^2 \theta}{8\pi^2} \right)^2. \quad (19)$$

Replacing (10) with (11) and using

$$\begin{aligned} & f_{\mu\nu} \left[-\frac{1}{2g^2} \mathcal{C}^{\mu\nu} + \frac{\theta}{16\pi^2} \tilde{\mathcal{C}}^{\mu\nu} \right] \\ & + \frac{1}{2} B_{\mu\nu} \left[c_0 \mathcal{C}^{\mu\nu} + c_1 \tilde{\mathcal{C}}^{\mu\nu} \right] \\ & = -\frac{1}{2g^2} A^{\mu a} \left\{ \epsilon^{ab3} f_{\mu\nu} - \frac{g^2 \theta}{8\pi^2} \epsilon^{ab3} \tilde{f}_{\mu\nu} \right. \\ & \quad \left. - g^2 c_0 \epsilon^{ab3} B_{\mu\nu} - g^2 c_1 \epsilon^{ab3} \tilde{B}_{\mu\nu} \right\} A^{\nu b}, \end{aligned} \quad (20)$$

the total action reads

$$\begin{aligned} & \mathcal{S}_{YM\theta}[\mathcal{A}] \\ & = \int_x \left\{ -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \frac{\theta}{32\pi^2} f_{\mu\nu} \tilde{f}^{\mu\nu} \right. \\ & \quad \left. - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{2g^2} A^{\mu a} Q_{\mu\nu}^{ab} A^{\nu b} \right\}, \end{aligned} \quad (21)$$

where we have defined

$$\begin{aligned} Q_{\mu\nu}^{ab} & := (D_\rho[a] D_\rho[a])^{ab} \delta_{\mu\nu} - 2\epsilon^{ab3} f_{\mu\nu} \\ & + g^2 c_1 \epsilon^{ab3} \tilde{B}_{\mu\nu} + g^2 c_0 \epsilon^{ab3} B_{\mu\nu} \\ & - D_\mu[a]^{ac} D_\nu[a]^{cb}. \end{aligned} \quad (22)$$

2.2 Gauge fixing

We adopt the gauge fixing (GF) condition for the off-diagonal part:

$$F^\pm[A, a] \equiv (\partial^\mu \pm i\xi a^\mu) A_\mu^\pm = 0, \quad (23)$$

where we have used the $(\pm, 3)$ basis, $\mathcal{O}^\pm \equiv (\mathcal{O}^1 \pm i\mathcal{O}^2)/\sqrt{2}$. Here the gauge parameter $\xi = 0$ corresponds to the Lorentz gauge and $\xi = 1$ to (the differential form of) the maximal abelian gauge (MAG). At this stage, we keep the residual U(1) gauge invariance without fixing it.

In the BRST quantization, the GF condition (23) amount to adding the following GF term and the Faddeev–Popov (FP) term [10],

$$\begin{aligned} \mathcal{L}_{GF+FP} & = \phi^a F^a[A, a] + \frac{\alpha}{2} (\phi^a)^2 \\ & + i\bar{c}^a D^{\mu ab}[a]^\xi D_\mu^{bc}[a] c^c \\ & - i\xi \bar{c}^a [A_\mu^a A^{\mu b} - A_\mu^c A^{\mu c} \delta^{ab}] c^b, \end{aligned} \quad (24)$$

where

$$\begin{aligned} F^a[A, a] & = (\partial^\mu \delta^{ab} - \xi \epsilon^{ab3} a^\mu) A_\mu^b \\ & = D^{\mu ab}[a]^\xi A_\mu^b. \end{aligned} \quad (25)$$

Thus the total Lagrangian is obtained by adding (21) to (24),

$$\mathcal{L} = \mathcal{L}_{apYM}[\mathcal{A}, \theta] + \mathcal{L}_{GF+FP}. \quad (26)$$

2.3 Integration over all SU(2)/U(1) components

Now we integrate out the off-diagonal fields, ϕ^a , A_μ^a , c^a , \bar{c}^a belonging to SU(2)/U(1) and obtain the Abelian-projected effective gauge theory (APEGT) written in terms of the diagonal fields, a_μ and $B_{\mu\nu}$.

Integrating the Lagrange multiplier field ϕ^a
For $\alpha \neq 0$, the Gaussian integration over ϕ^a can be done with ease as

$$\begin{aligned} & \phi^a F^a[A, a] + \frac{\alpha}{2} (\phi^a)^2 \\ & \rightarrow -\frac{1}{2\alpha} (F^a[A, a])^2. \end{aligned} \quad (27)$$

Integrating the off-diagonal gluon field A_μ^a
For the gauge parameter $\alpha = 1$, the total action

in the MAG ($\xi = 1$) reads[†]

$$\begin{aligned} S_{YM\theta} &= S_{YM\theta}[a, A, B, c, \bar{c}; \theta] \\ &= S_1[a, B; \theta] + S_2[a, c, \bar{c}] \\ &\quad + S_3[a, A, B, c, \bar{c}; \theta], \end{aligned} \quad (28)$$

$$\begin{aligned} S_1 &= \int_x \left[-\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \frac{\theta}{32\pi^2} f_{\mu\nu} \tilde{f}^{\mu\nu} \right. \\ &\quad \left. - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} \right], \end{aligned} \quad (29)$$

$$S_2 = \int_x i \bar{c}^a D^{\mu ac} [a] D_\mu^{cb} [a] c^b, \quad (30)$$

$$S_3 = \int_x \frac{1}{2g^2} A^{\mu a} Q_{\mu\nu}^{ab} A^{\nu b}, \quad (31)$$

$$\begin{aligned} Q_{\mu\nu}^{ab} &\equiv (D_\rho [a] D_\rho [a])^{ab} \delta_{\mu\nu} - 2\epsilon^{ab3} f_{\mu\nu} \\ &\quad + g^2 c_1 \epsilon^{ab3} \tilde{B}_{\mu\nu} + g^2 c_0 \epsilon^{ab3} B_{\mu\nu} \\ &\quad - 2ig^2 (\bar{c}^a c^b - \bar{c}^c c^c \delta^{ab}) \delta_{\mu\nu}, \end{aligned} \quad (32)$$

where we have rescaled the gauge parameter α such that $g^2/\alpha \rightarrow 1/\alpha$ and completed the square for the field A_μ . The off-diagonal gluon field A_μ^a in S_3 can be eliminated by the Gaussian integration,

$$e^{iS_0[a, B, c, \bar{c}; \theta]} = \int [dA_\mu^a] e^{iS_3[a, A, B, c, \bar{c}; \theta]}, \quad (33)$$

$$\begin{aligned} S_0 &= -i \ln \int [dA_\mu^a] \exp \left\{ i \int_x \frac{1}{2g^2} A^{\mu a} Q_{\mu\nu}^{ab} A^{\nu b} \right\} \\ &= \frac{i}{2} \ln \det(Q_{\mu\nu}^{ab}). \end{aligned} \quad (34)$$

Thus we obtain the APEG T with θ -term,

$$\begin{aligned} S_E &= S_0[a, B, c, \bar{c}; \theta] + S_1[a, B; \theta] \\ &\quad + S_2[a, c, \bar{c}]. \end{aligned} \quad (35)$$

2.4 Calculating the $\ln \det Q$ (Euclidean)

The $\ln \det Q$ in (34) is divergent. To regularize it, we use the ζ -function regularization [10]:

$$\ln \det Q = - \lim_{s \rightarrow 0} \frac{d}{ds} \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr}(e^{-tQ}). \quad (36)$$

[†]We can introduce the source term $\mathcal{A}_\mu J^\mu$ with the source J^μ to obtain the generating functional as in [10].

We evaluate $\text{Tr}(e^{-tQ})$ in the Euclidean space: $x_0 = -i\bar{x}_0$, $d^4x = -id^4\bar{x}$, to obtain

$$\begin{aligned} &\text{Tr}(e^{-tQ}) - \text{Tr}(e^{-t\partial^2 \delta^{ab} \delta_{\mu\nu}}) \\ &= \frac{1}{16\pi^2} \int_{\bar{x}} \text{tr} \left(\frac{1}{2} \tilde{Q}^2 + \frac{1}{12} [D_\mu, D_\nu] [D_\mu, D_\nu] \right) \\ &\quad + O(t), \end{aligned} \quad (37)$$

where

$$\begin{aligned} \tilde{Q}_{\mu\nu}^{ab} &\equiv Q_{\mu\nu}^{ab} - (D_\rho [a] D_\rho [a])^{ab} \delta_{\mu\nu} \\ &= -2\epsilon^{ab3} f_{\mu\nu} + g^2 c_1 \epsilon^{ab3} \tilde{B}_{\mu\nu} + g^2 c_0 \epsilon^{ab3} B_{\mu\nu} \\ &\quad - 2ig^2 (\bar{c}^a c^b - \bar{c}^c c^c \delta^{ab}) \delta_{\mu\nu}. \end{aligned} \quad (38)$$

By taking into account (8), the trace of the second term in (37) reads

$$\text{tr} \left(\frac{1}{12} [D_\mu, D_\nu] [D_\mu, D_\nu] \right) = -\frac{1}{3} \kappa f_{\mu\nu} f_{\mu\nu}, \quad (39)$$

where we have introduced the second Casimir operator κ which is given for $G = SU(2)$ by

$$\kappa \equiv C_2(G) = \epsilon^{3ab} \epsilon^{3ab} = 2. \quad (40)$$

The trace of the first term in (37) reads

$$\begin{aligned} \text{tr} \left(\frac{1}{2} \tilde{Q}^2 \right) &= \frac{1}{2} \tilde{Q}_{\mu\nu}^{ab} \tilde{Q}_{\nu\mu}^{ba} \\ &= 2\kappa f_{\mu\nu} f_{\mu\nu} - \frac{1}{2} g^4 \kappa B_{\mu\nu} B_{\mu\nu} \\ &\quad - 8g^4 (\bar{c}^a c^b - \bar{c}^c c^c \delta^{ab}) (\bar{c}^b c^a - \bar{c}^d c^d \delta^{ba}) \\ &\quad - 2\kappa g^2 c_1 B_{\mu\nu} \tilde{f}_{\mu\nu} - 2\kappa g^2 c_0 f_{\mu\nu} B_{\mu\nu} \\ &\quad + g^4 \frac{g^2 \theta}{16\pi^2} \kappa \tilde{B}_{\mu\nu} B_{\mu\nu}, \end{aligned} \quad (41)$$

where we have used the relationship in the Euclidean space, $c_1^2 + c_0^2 = -1$ and $c_0 c_1 = g^2 \theta / (16\pi^2)$.

Hence, (36) reads

$$\begin{aligned} &\frac{1}{2} \ln \det Q \\ &= \int_{\bar{x}} \left[\frac{1}{4g^2} z_a f_{\mu\nu} f_{\mu\nu} + \frac{1}{4} z_b g^2 B_{\mu\nu} B_{\mu\nu} \right. \\ &\quad \left. + \frac{1}{2} z_c B_{\mu\nu} \tilde{f}_{\mu\nu} + \frac{1}{2} z_d f_{\mu\nu} B_{\mu\nu} + \frac{1}{2} z_e B_{\mu\nu} \tilde{B}_{\mu\nu} \right. \\ &\quad \left. + (4\text{-ghost terms}) \right. \\ &\quad \left. + (\text{higher derivative terms}) \right], \end{aligned} \quad (42a)$$

where

$$z_a = -\frac{10}{3}\kappa\frac{g^2}{16\pi^2}\ln\mu^2, \quad (42b)$$

$$z_b = \kappa\frac{g^2}{16\pi^2}\ln\mu^2, \quad (42c)$$

$$z_c = 2\kappa c_1\frac{g^2}{16\pi^2}\ln\mu^2, \quad (42d)$$

$$z_d = 2\kappa c_0\frac{g^2}{16\pi^2}\ln\mu^2, \quad (42e)$$

$$z_e = -\kappa\frac{g^4}{16\pi^2} \cdot \frac{g^2\theta}{16\pi^2}\ln\mu^2. \quad (42f)$$

Note that $z_d, z_e \rightarrow 0$ as $\theta \rightarrow 0$. This result shows that the $f_{\mu\nu}f^{\mu\nu}$ and $B_{\mu\nu}B^{\mu\nu}$ terms receive no corrections to one-loop order coming from the existence of the θ term.

Going back to the Minkowski spacetime by taking into account $\int d^4\bar{x} = i \int d^4x$ in (42a), we obtain

$$\begin{aligned} & \frac{i}{2} \ln \det Q \\ &= - \int_x \left[\frac{1}{4g^2} z_a f_{\mu\nu} f^{\mu\nu} + \frac{1}{4} z_b g^2 B_{\mu\nu} B^{\mu\nu} \right. \\ &+ \frac{1}{2} z_c B_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{1}{2} z_d f_{\mu\nu} B^{\mu\nu} + \frac{1}{2} z_e B_{\mu\nu} \tilde{B}^{\mu\nu} \\ &+ (4\text{-ghost terms}) \\ &+ \left. (\text{higher derivative terms}) \right]. \end{aligned} \quad (43)$$

Thus we obtain

$$\begin{aligned} & S_0 + S_1 \\ &= \int_x \left[-\frac{1+z_a}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1+z_b}{4} g^2 B_{\mu\nu} B^{\mu\nu} \right. \\ &- \frac{1}{2} z_c B_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{\theta}{32\pi^2} f_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{1}{2} z_d f_{\mu\nu} B^{\mu\nu} \\ &+ \frac{1}{2} z_e B_{\mu\nu} \tilde{B}^{\mu\nu} + (4\text{-ghost terms}) \\ &+ \left. (\text{higher derivative terms}) \right]. \end{aligned} \quad (44)$$

Integrating the off-diagonal ghost field c^a, \bar{c}^a

Neglecting the 4-ghost term and the higher derivative terms in (44), the integration over c^a and \bar{c}^a in (30) reduces to the Gaussian type and can be performed easily [10]. The result to one-loop order reads

$$\begin{aligned} iS_c &= \ln \int [d\bar{c}][dc] \exp \left\{ i \int_x i\bar{c}^a D^{\mu ac}[a] D_\mu^{cb}[a] c^b \right\} \\ &= i \int_x \frac{1}{4g^2} z'_a f_{\mu\nu} f^{\mu\nu} + \dots, \end{aligned} \quad (45a)$$

where

$$z'_a = \frac{1}{3}\kappa\frac{g^2}{16\pi^2}\ln\mu^2. \quad (45b)$$

3 APEGT with dyon

The U(1) antisymmetric tensor field $B_{\mu\nu}$ has the Hodge-de Rham decomposition,

$$B_{\mu\nu} = b_{\mu\nu} + \tilde{\chi}_{\mu\nu}, \quad (46a)$$

$$b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu, \quad (46b)$$

$$\tilde{\chi}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}(\partial^\alpha \chi^\beta - \partial^\beta \chi^\alpha). \quad (46c)$$

Then the integration measure for $B_{\mu\nu}$ is replaced by the measure for b_μ and χ_μ ,

$$[dB_{\mu\nu}] = [db_\mu][d\chi_\mu]\delta(F[b])\delta(F[\chi]), \quad (47)$$

where $F[b]$ and $F[\chi]$ are gauge fixing conditions for the gauge symmetries,

$$\begin{aligned} b_\mu(x) &\rightarrow b_\mu(x) - \partial_\mu \theta(x), \\ \chi_\mu(x) &\rightarrow \chi_\mu(x) - \partial_\mu \varphi(x). \end{aligned} \quad (48)$$

Thus, (44) reads

$$\begin{aligned} & S_0 + S_1 \\ &= \int_x \left[-\frac{1+z_a}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1+z_b}{4} g^2 (b_{\mu\nu} b^{\mu\nu} \right. \\ &+ \tilde{\chi}_{\mu\nu} \tilde{\chi}^{\mu\nu}) - \frac{1}{2} z_c b_{\mu\nu} \tilde{f}^{\mu\nu} - \frac{1}{2} z_c \tilde{\chi}_{\mu\nu} \tilde{f}^{\mu\nu} \\ &+ \frac{1}{4} \frac{\theta}{8\pi^2} f_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{1}{2} z_d f_{\mu\nu} b^{\mu\nu} + \frac{1}{2} z_d f_{\mu\nu} \tilde{\chi}^{\mu\nu} \\ &+ \left. \frac{1}{2} z_e b_{\mu\nu} \tilde{b}^{\mu\nu} + z_e b_{\mu\nu} \chi^{\mu\nu} \right]. \end{aligned} \quad (49)$$

Here integrating out the χ_μ field yields the corrections to $f f$ -, $f \tilde{f}$ -, $b b$ -terms. However, they do not affect the one-loop results. Therefore, the effective action to one-loop order is obtained by taking into account the contribution (45b) from the ghost as

$$\begin{aligned} S_E &= \int_x \left[-\frac{1+z_a-z'_a}{4g^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{4} \frac{\theta}{8\pi^2} f_{\mu\nu} \tilde{f}^{\mu\nu} \right. \\ &- \frac{1+z_b}{4} g^2 b_{\mu\nu} b^{\mu\nu} - \frac{1}{2} z_c b_{\mu\nu} \tilde{f}^{\mu\nu} \\ &+ \left. \frac{1}{2} z_d f_{\mu\nu} b^{\mu\nu} + \frac{1}{2} z_e b_{\mu\nu} \tilde{b}^{\mu\nu} \right]. \end{aligned} \quad (50a)$$

Here the last two terms in (50a) are cast into

$$f_{\mu\nu}b^{\mu\nu} = -2\partial_\nu b_\mu f^{\mu\nu} = 2b_\mu \partial_\nu f^{\mu\nu} + (\text{surface term}), \quad (50b)$$

$$b_{\mu\nu}\tilde{b}^{\mu\nu} = 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu b_\nu \partial_\rho b_\sigma = -2\epsilon^{\mu\nu\rho\sigma}b_\nu \partial_\mu \partial_\rho b_\sigma + (\text{surface term}). \quad (50c)$$

and they are neglected, provided that $\partial_\nu f^{\mu\nu} = J^\mu = 0$ and b_μ is regular.

Defining the magnetic current k_μ by

$$k^\mu \equiv \partial_\nu \tilde{f}^{\mu\nu}, \quad \tilde{f}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f_{\rho\sigma}, \quad (51)$$

we obtain the APEGT including the magnetic current k_μ from (50a)

$$S_E = \int_x \left[-\frac{1}{4}Z_a^{-1}f_{\mu\nu}f^{\mu\nu} + \frac{g\theta}{16\pi^2}a_\mu k^\mu - \frac{1}{4}Z_b^{-1}b_{\mu\nu}b^{\mu\nu} - \frac{1}{2g}z_cb_\mu k^\mu \right], \quad (52)$$

where we have defined

$$Z_a \equiv (1 + z_a - z'_a)^{-1} = 1 - z_a + z'_a, \quad (53)$$

$$Z_b \equiv (1 + z_b)^{-1} = 1 - z_b,$$

and rescaled $a_\mu/g \rightarrow a_\mu$ and $gb_\mu \rightarrow b_\mu$.

We define the wave function renormalization for a_μ and b_μ by

$$a_\mu^R \equiv Z_a^{-1/2}a_\mu, \quad b_\mu^R \equiv Z_b^{-1/2}b_\mu. \quad (54)$$

Then (52) is cast into the renormalized form,

$$S_E = \int_x \left[-\frac{1}{4}f_{\mu\nu}^R f^{R\mu\nu} - \frac{1}{4}b_{\mu\nu}^R b^{R\mu\nu} + \frac{g\theta^R}{2\pi}a_\mu^R k^{R\mu} - \frac{4\pi}{g^R}b_\mu^R k^{R\mu} \right], \quad (55)$$

where we have defined the renormalized quantities,

$$g^R \equiv Z_a^{1/2}g, \quad (56a)$$

$$\theta^R \equiv Z_a^{-1/2}Z_c^{-1}Z_b^{-1/2}\theta, \quad (56b)$$

$$k^{R\mu} \equiv Z_a^{1/2}Z_cZ_b^{1/2}k^\mu, \quad (56c)$$

and rescaled $k_\mu^R \rightarrow 8\pi k_\mu^R$. We find that k_μ has the renormalization factor Z_a due to the existence of the θ term. The existence of the third

term, the cross term of the magnetic current k_μ with the electric field a_μ , in (55) indicates that *the monopole current k_μ acquires the electric charge and the magnetic monopole is changed to the dyon due to the existence of the θ term* in agreement with the Witten effect [13].[§] In what follows, we omit the index R of the field.

We observe that the Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}b_{\mu\nu}b^{\mu\nu} \quad (57)$$

is invariant under the linear transformation for a_μ and b_μ with an arbitrary constant v :

$$\begin{pmatrix} a_\mu \\ b_\mu \end{pmatrix} = \begin{pmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{pmatrix} \begin{pmatrix} a'_\mu \\ b'_\mu \end{pmatrix}. \quad (58)$$

By choosing

$$v = \arctan\left(-\frac{g^2\theta}{8\pi^2}\right), \quad (59)$$

we can eliminate the cross term in (55) which is transformed into

$$S_E = \int_x \left[-\frac{1}{4}f'_{\mu\nu}f'^{\mu\nu} - \frac{1}{4}b'_{\mu\nu}b'^{\mu\nu} - g_m[\theta]b'_\mu k'^\mu \right], \quad (60)$$

where

$$g_m[\theta] := g|\tau| = \sqrt{g_m^2 + q_m^2},$$

$$g_m \equiv \frac{4\pi}{g}, \quad q_m \equiv \frac{g\theta}{2\pi}. \quad (61)$$

Finally, by integrating out the field b'_μ , we obtain (omitting the prime in what follows)

$$S_E = \int_x \left[-\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2}g_m^2[\theta]k^\mu D_{\mu\nu}k^\nu \right], \quad (62)$$

where the kernel $D_{\mu\nu}$ stands for the massless vector propagator obtained after introducing the gauge fixing term for b'_μ , e.g., $D_{\mu\nu} = (1/\partial^2)(\delta_{\mu\nu} - \partial_\mu \partial_\nu / \partial^2)$ in the Landau gauge. It is remarkable that the effect of the θ angle is combined into a compact form written in terms

[§]The dyon treated in this paper goes to the usual magnetic monopole in the vanishing θ angle limit, $\theta \rightarrow 0$. Therefore, it is different from the usual (old-fashioned) dyon [14, 15] which is the magnetic monopole having a non-vanishing electric charge even for $\theta = 0$.

of the complex coupling constant τ even after the Abelian projection, since $g_m[\theta] = g|\tau|$.

This result should be compared with the effective theory (3.4) of [9] written in terms of the electric current j_μ and the magnetic current k_μ . Indeed, if the electric current j_μ is eliminated in (3.4) of [9], the resulting theory agrees with our result (62). However, it was assumed in [9] that the Yang-Mills theory is approximated in terms of Abelian fields with the θ angle at a long-distance scale R .

4 Topological susceptibility and Witten-Veneziano formula

Now we argue that *the dyon configuration is the most relevant one for solving the $U(1)$ problem in $SU(2)$ QCD* by evaluating the topological susceptibility from the dyon configuration appearing in the APEGT with θ -term.

Integrating out a_μ in (62), we obtain the *effective dyon action*

$$S_E = \int_x \left[\frac{1}{2} \left\{ \left(\frac{4\pi}{g} \right)^2 + \left(\frac{g\theta}{2\pi} \right)^2 \right\} k^\mu D_{\mu\nu} k^\nu \right]. \quad (63)$$

To estimate the numerical value of the topological susceptibility, we consider the lattice regularized version of (63),

$$S_E = \sum_{x,y} \left(\bar{\beta} + \frac{\theta^2}{\bar{\beta}} \right) k^\mu(x) D_{\mu\nu}(x-y) k^\nu(y),$$

$$\bar{\beta} \equiv \frac{1}{2} \left(\frac{4\pi}{g} \right)^2. \quad (64)$$

The part of the self-mass term $k^\mu(x)k^\mu(x)$ is extracted from (64) as[¶]

$$S_E \simeq \left(\bar{\beta} + \frac{\theta^2}{\bar{\beta}} \right) D(0) \sum_x k^\mu(x) k^\mu(x), \quad (65)$$

[¶]According to the analysis of the monopole action by the inverse Monte-Carlo simulation, the self-mass term of the monopole current is dominant in the low-energy region, e.g., $G_2/G_1 \simeq 0.33$ at the scale 1.7fm where G_1 and G_2 are respectively the self-coupling and the nearest-neighbor coupling of the monopole current [19].

where $D(0) < \infty$ on a lattice. Furthermore, as the monopole configuration subject to^{||} $|k_\mu(x)| = 1$ is dominant in the low-energy region [17], the energy density e_θ is written as

$$e_\theta = S_E/V \simeq \left(\bar{\beta} + \frac{\theta^2}{\bar{\beta}} \right) D(0). \quad (66)$$

Therefore, the topological susceptibility χ_E is calculated:^{**}

$$\chi_E \equiv \left(\frac{d^2 e_\theta}{d\theta^2} \right)_{\theta=0} \simeq \frac{2}{\bar{\beta}} D(0). \quad (67)$$

The result of Chernodub et al.[19] show $\bar{\beta}D(0) \equiv G_1 = 0.059$ and $\bar{\beta} = 2.49$ at the physical scale $b = 3.8\sigma_{phys}^{-1/2}$. (Note that $b = 1\sigma_{phys}^{-1/2}$ corresponds to 1.7fm, provided that the string tension $\sigma_{phys} \cong (440\text{MeV})^2$ in $SU(2)$ QCD.) By substituting these values into (67), the topological susceptibility is determined as^{††}

$$\chi_E^{1/4}/\sigma_{phys}^{1/2} = 0.371, \quad (68)$$

in units of the string tension σ_{phys} . Remarkably, this estimate reproduces 76% of the full result

$$\chi^{1/4}/\sigma_{phys}^{1/2} = 0.486 \pm 0.010, \quad (69)$$

obtained by Teper [16] in the simulation of $SU(2)$ QCD. Moreover, our result is also consistent with those of Bornyakov and Schierholz [21] and Sasaki and Miyamura [22] where the Abelian dominance for χ was reported based on the numerical simulations.

For large N_c in the $SU(N_c)$ QCD, Witten [5] has shown by taking into account the next-to-leading order of $1/N_c$ expansion that the $U(1)$ chiral symmetry is broken due to the axial-vector

^{||}The monopole current is integer-valued on the lattice, if the construction due to DeGrand and Toussaint [18] is used.

^{**}Here θ should be understood as the renormalized variable θ_R .

^{††}In this section, we have used a quantum perfect monopole action to evaluate the topological susceptibility χ_E , that is an action on the renormalized trajectory on which one can take the continuum limit. Therefore, our prediction agree with those of the continuum independently whether the lattice is fine or coarse. See [20] for the detail of the quantum perfect action.

anomaly and hence the NG boson η' can acquire the non-zero mass of $\mathcal{O}(N_c^{-1})$ even in the chiral limit. Moreover, Witten has derived the mass formula for η' , the so-called the Witten-Veneziano formula[6]:

$$m_{\eta'}^2 = \frac{4N_f}{f_\pi^2} \left(\frac{d^2 E_\theta}{d\theta^2} \right)_{\theta=0}^{\text{no-quarks}}, \quad (70)$$

where E_θ is the vacuum energy density of the gluon field. Substituting the numerical values, $m_{\eta'} \simeq 1\text{GeV}$ and $f_\pi \simeq 0.1\text{GeV}$ into (70), the topological susceptibility is estimated as (for $N_f = 3$)

$$\begin{aligned} \chi &\equiv \left(\frac{d^2 E_\theta}{d\theta^2} \right)_{\theta=0}^{\text{no-quarks}} \\ &= \frac{1}{12} (0.1\text{GeV})^2 (1\text{GeV})^2 \\ &\simeq 8 \times 10^{-4} (\text{GeV})^4, \\ &\rightarrow \chi^{1/4} \simeq 150 \sim 200 \text{MeV}. \end{aligned} \quad (71)$$

This result for large N_c of $SU(N_c)$ case is consistent with the $SU(2)$ result (69). In order for this formula to be meaningful, the vacuum energy density E_θ must depend on the θ angle.

Although we have restricted to the $SU(2)$ gauge group in this analysis, these results suggest that the large N_c analysis in the next-to-leading order gives fairly good estimate also for $N_c = 2$ in the $U(1)$ problem. In fact, this claim is also confirmed by the numerical simulations on the lattice for various N_c of $SU(N_c)$ Yang-Mills theory, see Teper [16].

Thus we conclude that *the dyon, i.e., magnetic monopole with the electric charge proportional to the vacuum angle θ , gives dominant contribution to the topological susceptibility.*^{††}

5 Conclusion and discussion

In this paper, we have argued an interesting possibility that the $U(1)$ problem is solved by the

^{††}Note that the above estimate of the topological susceptibility was obtained based on the self-mass term alone. It is expected therefore that the inclusion of the remaining terms such as the nonlocal interaction terms reproduce the whole topological susceptibility of the original $SU(2)$ gluodynamics.

dyonic configuration appearing in the APEGT with a vacuum angle θ . For this purpose, we started with the Yang-Mills theory with a vacuum angle θ in the MAG. We have separated the Abelian component by exploiting the idea of the Abelian projection and then integrated out all the off-diagonal components except for the diagonal ones. Applying a duality transformation to the resulting theory, we have obtained an effective theory written in terms of the dyon degrees of freedom, called the APEGT with θ -term. By making use of the classical part of the dyon action, we have estimated the topological susceptibility. The obtained value agrees with the numerical result obtained by the recent lattice gauge theory. Thus we have shown that the dyon configuration generated by the vacuum angle θ gives a dominant contribution to the topological susceptibility and resolves the $U(1)$ problem.

In this paper, we have treated only the $SU(2)$ case in detail. In order to confirm the consistency of our claim with the large N_c result, it is desirable to extend our method to $SU(3)$ case. This will be reported in a subsequent paper [25].

In the derivation above, there is a subtle point to be mentioned. To estimate the topological susceptibility, we have translated the continuum result (63) to the lattice one (64), and used a fact that $D(0)$ is finite on the lattice. However, the kernel $D_{\mu\nu}$ in the continuum (63) does not have the contact interaction, and hence the self-mass term of the monopole current does not exist in the rigorous sense within this derivation. Therefore, it should be understood that we have introduced a cutoff in (65) to regularize $D_{\mu\nu}(x-y)$ so that $D(0) < \infty$, just as Ezawa and Iwazaki [9] replaced $D_{\mu\nu}(x-y)$ by the massless propagator with the momentum cutoff R^{-1} .

Such a physical cutoff naturally appears if the off-diagonal gluons acquire their mass. In fact, the numerical simulations on the lattice have confirmed the non-zero mass for the off-diagonal gluons [11]. Some analytical studies in this direction exist too, see [23]. It is possible to show that the self-interaction term between monopole currents and the derivative term appear as a consequence of mass generation [24]. This point

deserves further studies in connection with the U(1) problem [25].

We comment on the role of instantons to the U(1) problem. In this paper we have stressed that the magnetic monopole promoted to the dyon due to the vacuum angle θ resolves the U(1) problem. Our results suggest the compatibility between the dyon configuration with the large N_c expansion, although it is not definitive. Although instantons and magnetic monopole are originated from different non-trivial homotopy groups, $\pi_3(SU(N_c)) = \mathbb{Z}$ and $\pi_2(SU(N_c)/U(1)^{N_c-1}) = \mathbb{Z}^{N_c-1}$ respectively [26], the strong correlation between instantons and magnetic monopoles have been reported recently for $N_c = 2$, see e.g., [27] for analytical works and [28] for numerical works. Moreover, there is a vast and consistent literature concerning the lattice determination of the pure-gauge topological susceptibility on the lattice, in which the role of instantons for $N_c = 2$ and $N_c = 3$ has been well proved and tested by means of the so-called *cooling method*, see e.g., [29] for SU(2), [30] for SU(3) and [31] for a review. The large- N_c behaviour of instantons is not so clear and far from definite, as discussed already in the classic papers [32]. Thus we arrive at a viewpoint that the magnetic monopole and the dyon should be treated on an equal footing with the instanton as non-perturbative topological configurations to be taken account of in solving the U(1) problem.

In this paper, we have discussed the U(1) problem only within the framework of the pure Yang-Mills theory with the vacuum angle θ . Suppose that the quark degrees of freedom are introduced into the consideration. Then it is interesting to study the relationship of our results with the Atiyah–Singer index theorem [34] (see also chapter 11 of [35]), the fermionic zero modes (low-lying eigenvalue of the Dirac operator), and Banks–Casher formula [33]. Such investigations have already been done on a lattice by numerical simulations, e.g., see [36] in the instanton background and [37] in the monopole background. These results also demonstrate the strong correlation between instantons and magnetic monopoles. The investigation from the an-

alytical side is a future task.

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